Quadrature Formula for a Double-Pole Singular Integral

Rajendra K. Bera*
National Aeronautical Laboratory, Bangalore, India

Introduction

IN linearized potential thin-wing and airfoil theory, one encounters integrals of the following two forms:

$$I_1(x) = \int_{-1}^{1} (\xi - x)^{-1} f(\xi)\, d\xi, \qquad -1 < x < +1 \tag{1}$$

$$I_2(x) = \int_{-1}^{1} (\xi - x)^{-2} f(\xi)\, d\xi, \qquad -1 < x < +1 \tag{2}$$

In the conventional (Riemann) sense, these integrals are meaningless because of the pole singularity at $x = \xi$. For the origin and context of these important integrals, one may refer to Mangler,[1] who also provides rules for their appropriate interpretation.

The value of these integrals, when they exist, is called the principal value of the concerned integral, when they are interpreted, respectively, in the following limiting sense:

$$\int_{-1}^{1} (\xi - x)^{-1} f(\xi)\, d\xi = \lim_{\epsilon \to 0} \left[\int_{-1}^{x-\epsilon} (\xi - x)^{-1} f(\xi)\, d\xi \right.$$

Received Nov. 18, 1987.

*Scientist, Fluid Mechanics Division.

$$+ \int_{x+\epsilon}^{1} (\xi - x)^{-1} f(\xi)\, d\xi \Bigg] \tag{3}$$

$$\int_{-1}^{1} (\xi - x)^{-2} f(\xi)\, d\xi = \lim_{\epsilon \to 0} \Bigg[\int_{-1}^{x-\epsilon} (\xi - x)^{-2} f(\xi)\, d\xi$$

$$+ \int_{x+\epsilon}^{1} (\xi - x)^{-2} f(\xi)\, d\xi - 2f(x)/\epsilon \Bigg] \tag{4a}$$

One can show that in those cases where an indefinite integral can be found, the answer can be found by simply inserting the limits $\xi = -1$ and $\xi = +1$, provided any logarithm of $(\xi - x)$ that appears is interpreted as $ln\,|\,\xi - x\,|$. This important property, in effect, allows the use of conventional integration techniques in many problems. For example, one may write Eq. (4a) in the alternative form

$$\int_{-1}^{1} (\xi - x)^{-2} f(\xi)\, d\xi = \frac{d}{dx} \int_{-1}^{1} (\xi - x)^{-1} f(\xi)\, d\xi \tag{4b}$$

However, there are many other problems where numerical integration of these integrals must be used.

In this brief Note, we show that the generalized quadrature formula derived by Stark[2] for the single-pole singular integral $I_1(x)$ can be extended in a simple way to evaluate the double-pole singular integral $I_2(x)$. The advantage of Stark's formula for $I_1(x)$ is that it is tailormade for integrands containing a weight function $W(x)$ such that $f(x)$ is the product of a regular function and the weight function $W(x)$, where $W(x)$ is assumed to be positive and integrable but not necessarily regular. This advantage is carried over to the extension of Stark's formula for $I_2(x)$.

II. Stark's Formula

Stark[2] provided the following Gaussian quadrature formula for $I_1(x)$ having the weight function $W(x)$:

$$\int_{-1}^{1} (\xi - x)^{-1} f(\xi)\, d\xi = \sum_{i=1}^{N} e_i f(\xi_i)\, (\xi_i - x_j)^{-1} \tag{5}$$

which is exactly valid at the points $x = x_j$ noted subsequently under the conditions that 1) the ratio $f(x)/W(x)$ is a polynomial of degree $\leq 2N$, 2) the points $x = \xi_i$, $i = 1, 2, \ldots, N$ are the N-zeros of the polynomial $P_N(x)$ of degree N in the system $\{P_i(x)\}$ of orthogonal polynomials, assuming $W(x)$ as the weight function in $-1 < x < +1$, 3) the points $x = x_j$, $j = 1, 2, \ldots$, are zeros of the function

$$Q_N(x) = -\frac{1}{2} \int_{-1}^{1} W(\xi) P_N(\xi) (\xi - x)^{-1} d\xi \tag{6}$$

and 4) the coefficients e_i are defined by

$$e_i = [W(\xi_i) P_N'(\xi_i)]^{-1} \int_{-1}^{1} W(\xi) P_N(\xi)(\xi - \xi_i)^{-1} d\xi \tag{7}$$

where $P_N'(\xi_i)$ is the derivative of $P_N(\xi)$ at $x = \xi_i$.

The weights e_i and the abscissas ξ_i are seen to be identical with those of the ordinary Gaussian quadrature formula for the weight function $W(x)$. Closed-form expressions for e_i, ξ_i, x_j for weight functions frequently used in thin-wing and airfoil theory (see Sec. IV) may be found in Ref. 3.

III. Extension of Stark's Formula to $I_2(x)$

For $I_2(x)$, we seek a quadrature formula for the weight function $W(x)$ in the form

$$\int_{-1}^{1} (\xi - x)^{-2} f(\xi) d\xi$$

$$= \sum_{i=1}^{N} e_i [1 - Q_N(x_j)/Q_N(\xi_i)] f(\xi_i)(\xi_i - x_j)^{-2} \tag{8}$$

This formula is exactly valid when $f(x)/W(x)$ is a polynomial of degree $\leq N$ at the points $x = x_j$, $j = 1, 2, \ldots$, provided x_j are now the zeros of the function $Q_N'(x)$, i.e.,

$$Q_N'(x) = -\frac{1}{2} \int_{-1}^{1} W(\xi) P_N(\xi)(\xi - x)^{-2} d\xi$$

$$= -\frac{1}{2} \frac{d}{dx} \int_{-1}^{1} W(\xi) P_N(\xi)(\xi - x)^{-1} d\xi = \frac{d}{dx} Q_N(x) \tag{9}$$

where ξ_i, e_i remain as before for the given weight function $W(x)$.

The proof is quite straightforward. Let

$$f_1(\xi) = \sum_{i=1}^{N} f(\xi_i) h_i(\xi) + A W(\xi) P_N(\xi) \tag{10}$$

where $f_1(\xi)$ is a test function such that $f_1(\xi)/W(\xi)$ is an arbitrary polynomial of degree N; A is a constant, and $h_i(\xi)$ are some interpolating functions defined by

$$h_i(\xi) = P_N(\xi) W(\xi) / [P_N'(\xi_i) W(\xi_i)(\xi - \xi_i)] \tag{11}$$

Equation (8) is valid if it is valid for $f_1(\xi)$, since $f_1(\xi)/W(\xi)$ is an arbitrary polynomial of degree N. Therefore, we substitute $f_1(\xi)$ for $f(\xi)$ on the left-hand side of Eq. (8) and obtain

$$L = \int_{-1}^{1} (\xi - x_j)^{-2} \sum_{i=1}^{N} f(\xi_i) h_i(\xi) d\xi$$

$$+ A \int_{-1}^{1} (\xi - x_j)^{-2} W(\xi) P_N(\xi) d\xi \tag{12}$$

The second integral is zero since it represents $Q_N'(x_j)$.

If we note that

$$(\xi_i - x_j)^2 (\xi - \xi_i)^{-1} (\xi - x_j)^{-2}$$

$$= (\xi - \xi_i)^{-1} - (\xi - x_j)^{-1} - (\xi_i - x_j)(\xi - x_j)^{-2} \tag{13}$$

we may, once again, using $Q_N'(x_j) = 0$, write

$$L = \int_{-1}^{1} (\xi - x_j)^{-2} f_1(\xi) d\xi$$

$$= \sum_{i=1}^{N} e_i [1 - Q_N(x_j)/Q_N(\xi_i)] f(\xi_i)(\xi_i - x_j)^{-2} \tag{14}$$

which completes the proof.

IV. Quadrature Formulas for $I_2(x)$ for Special Weight Functions

In thin-wing and airfoil theory, we frequently encounter the following weight functions: 1) $W(x) = (1 - x^2)^{-1/2}$, 2) $W(x) = (1 - x^2)^{1/2}$, 3) $W(x) = (1 - x)^{1/2} (1 + x)^{-1/2}$, 4) $W(x) = (1 + x)^{1/2} (1 - x)^{-1/2}$. From Ref. 3, their respective e_i, ξ_i, x_j for $I_1(x)$ are as shown in Table 1. The respective $P_N(x)$, $Q_N(x)$, which are all polynomials, are given in terms of the variable $\theta = \cos^{-1} x$, because their roots ξ_i and x_j can be obtained in closed form by simple inspection. The $P_N(x)$ and $Q_N(x)$ are related to Tchebycheff polynomials of the first and second kinds.

With the exception of $W(x) = (1 - x^2)^{1/2}$, for which

$$Q_N'(x) = (N+1) \frac{\pi}{2} \sin (N+1)\theta / \sin\theta$$

Table 1 Summary of quadrature formulas for $I_1(x)$

$W(x)$	$(1-x^2)^{-1/2}$	$(1-x^2)^{1/2}$	$\sqrt{(1-x)/(1+x)}$	$\sqrt{(1+x)/(1-x)}$
$P_N(x)$	$\cos N\theta$	$\sin(N+1)\theta/\sin\theta$	$C\sin(2N+1)(\theta/2)/\sin(\theta/2)$	$C\cos(2N+1)(\theta/2)/\cos(\theta/2)$
$Q_N(x)$	$-\frac{\pi}{2}\sin N\theta/\sin\theta$	$\frac{\pi}{2}\cos(N+1)\theta$	$-\frac{\pi}{2}C\cos(2N+1)(\theta/2)/\cos\theta$	$-\frac{\pi}{2}C\sin(2N+1)(\theta/2)/\sin(\theta/2)$
ξ_i	$\cos[(2i-1)\pi/2N]$, $i=1,2,\ldots,N$	$\cos[(i\pi/(N+1)]$, $i=1,2,\ldots,N$	$\cos[(2i\pi/(2N+1)]$, $i=1,2,\ldots,N$	$\cos[(2i-1)\pi/(2N+1)]$, $i=1,2,\ldots,N$
e_i	$\frac{\pi}{N}(1-\xi_i^2)^{1/2}$	$\frac{\pi}{N+1}(1-\xi_i^2)^{1/2}$	$\frac{2\pi}{2N+1}(1-\xi_i^2)^{1/2}$	$\frac{2\pi}{2N+1}(1-\xi_i^2)^{1/2}$
x_j	$\cos(j\pi/N)$, $j=1,2,\ldots,N-1$	$\cos[(2j-1)\pi/2(N+1)]$, $j=1,2,\ldots,N+1$	$\cos[(2j-1)\pi/(2N+1)]$, $j=1,2,\ldots,N$	$\cos[(2j\pi/(2N+1)]$, $j=1,2,\ldots,N$

$\theta=\cos^{-1}x$; $C=\Gamma(N+1/2)/N!\sqrt{\pi}$; $\Gamma(m)$ is the gamma function with argument m.

Table 2 Comparison of computed and exact solutions of
$$I_2(x)=\int_{-1}^{1}\frac{\xi^4 d\xi}{\sqrt{1-\xi^2}\,(\xi-x)^2},\ -1<x+1$$

x_j	Exact $I_2(x_j)$	Computed $I_2(x_j)$
	For $N=6$	
0.72741239	6.55771764	6.55771764
0.26621648	2.23874179	2.23874179
−0.26621648	2.23874179	2.23874179
−0.72741239	6.55771764	6.55771764

whose roots are

$$x_j=\cos[j\pi/(N+1)],\ j=1,2,\ldots,N$$

the roots of $Q_N'(x)$, unfortunately, do not seem to be extractable in closed form for an arbitrary value of N. Therefore, they must be determined numerically, say, by using the bisection method. We may note, however, that since $Q_N(x)$ is a polynomial, all of whose roots lie in the interval $-1<x<+1$, $Q_N'(x)$, which will be a polynomial of one degree less, will also have all of its roots in the interval $-1<x<+1$. Indeed, between two consecutive roots of $Q_N(x)$ will lie a root of $Q_N'(x)$.

V. Numerical Example of $I_2(x)$

For illustration, we choose to evaluate numerically the integral

$$I_2(x)=\int_{-1}^{1}\frac{\xi^4 d\xi}{\sqrt{1-\xi^2}(\xi-x)^2}=\pi(3x^2+0.5) \tag{15}$$

whose exact solution is also noted above. The $Q_N'(x)$ for the weight function $(1-x^2)^{-1/2}$ is

$$Q_N'(x)=-\frac{\pi}{4}[(N+1)\sin(N-1)\theta-(N-1)$$
$$\sin(N+1)\theta]/\sin^3\theta$$

where $\theta=\cos^{-1}x$. Let us choose $N=6$, for which we have

$$Q_6'(x)=-\pi(80x^4-48x^2+3)$$

The computed and exact values of $I_2(x)$ are shown in Table 2 and, as expected, they are identical. In passing, we note that integrands having any of the other three weight function, 2–4 in Sec. IV, can always be reinterpreted to have $(1-x^2)^{1/2}$ as its weight function.

VI. Conclusions

Stark's quadrature formula for Cauchy integrals has been extended in a simple manner to evaluate double-pole singular integrals, which occur in linear lifting surface theory.

References

[1]Mangler, K. W., "Improper Integrals in Theoretical Aerodynamics," British Aeronautical Research Council, London, R&M 2424, 1951.

[2]Stark, V. J. E., "A Generalised Quadrature Formula for Cauchy Integrals," *AIAA Journal*, Vol. 9, Sept. 1971, pp. 1854–1855.

[3]Bera, R. K., "The Numerical Evaluation of Cauchy Integrals," *International Journal of Mathematical Education in Science and Technology*, (to be published).